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On manifest $SU(4)$ invariant superstring action in $\mathrm{AdS}_5 \times \mathrm{S}^5$

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Abstract

We discuss manifestly $SL(2,C) \times SU(4)$ and κ invariant superstring action in $AdS_5 \times S^5$ background in the framework of Green-Schwarz formulation. The action is formulated in terms of 16 Poincaré fermionic coordinates which through AdS/CFT correspondence should represent $\mathcal{N} = 4$ SYM superspace and 16 superconformal fermionic coordinates. The action is also manifestly invariant with respect to the usual $\mathcal{N} = 4$ Poincaré superalgebra transformations. κ-symmetry gauge fixing and the derivation of light-cone gauge action is simplified.

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1 Introduction and summary of result

Motivated by the conjectured duality between the string theory and $\mathcal{N}=4$, $4d$ SYM theory [1] the Green-Schwarz formulation of strings propagating in $AdS_5\times S^5$ was suggested in [2] (for further developments see [3–6]). Alternative approaches based on exploiting twistor like variables can be found in [7–9]. Recently some progress in understanding light-cone gauge formulation has been achieved in [10, 11]. The action found in [10, 11] in the particle limit describes massless states of IIB supergravity and therefore it can be used to discuss sector of short fundamental strings. Alternative gauge fixed action found in [4–6], so called S-gauge action, describes sector of long strings [12, 13]; this action is suitable for discussion of various BPS configurations of AdS superstrings.

The original action of superstrings in $AdS_5\times S^5$ [2] being realized as sigma model on coset superspace $PSU(2,2|4)/SO(4,1)\times SO(5)$ has manifest symmetry with respect to $SO(4,1)\times SO(5)$. An interesting phenomenon discovered in [6, 10] is that after imposing κ-symmetry gauge the resulting action can be cast into manifestly $SU(4)$ invariant form. This fact while being expected for the bosonic part of action, surprisingly turns out to be also a feature of the fermionic sector of the GS action.

Here we would like to demonstrate that that not only gauge fixed actions have $SU(4)$ invariant formulation but the original κ-symmetric GS action [2] can be put into a manifestly $SU(4)$ invariant form. To derive κ and manifestly $SU(4)$ invariant superstring action in the framework of GS formulation is desirable for several reasons. Such κ-invariant action can be used to study interaction of superstring with $D3$ brane. As is well known, κ-symmetry plays defining role and can be used to fix interactions of superstring ending on $D3$ brane. The formulation we develop is based on splitting fermionic coordinates into θ and η. The variables θ represent the odd part of $\mathcal{N}=4$ Poincaré superalgebra superspace and they can be responsible for the description of boundary $\mathcal{N}=4$ SYM theory. The η are superconformal fermionic coordinates and they have nonlinear dynamics (even light-cone action involves terms of fourth degree in η). Because $\mathcal{N}=4$ SYM theory respects manifest $SU(4)$ invariance it is desirable to put superstring action into manifestly $SU(4)$ invariant form from the very beginning, i.e. at the level of κ-invariant action.

The superstring Lagrangian is formulated in terms of 10 bosonic coordinates (x^a, Z^M), (x^a are 4 coordinates along boundary directions, while Z^M represent one AdS radial coordinate and five coordinates of S^5) and 32 two fermionic coordinates $(\theta^{\mathsf{a}}_i, \theta^{\dot{\mathsf{a}}i}, \eta^i_{\mathsf{a}}, \eta^{\dot{\mathsf{a}}}_i)$ which transform in (anti)fundamental representations of $SU(4)$ with respect to indices i and one-half representations of $sl(2,C)$ with respect to indices $\mathsf{a}, \dot{\mathsf{a}}$. The Lagrangian is given by the sum of kinetic $\mathcal{L}_{kin}$ and Wess-Zumino terms

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{WZ}\,, \tag{1.1}$$

$$\mathcal{L}_{kin} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}(\hat{L}^{\mathsf{a}\dot{\mathsf{b}}}_\mu \hat{L}_{\nu\mathsf{a}\dot{\mathsf{b}}} + \frac{1}{Z^2}D_\mu Z^M D_\nu Z^M)\,, \tag{1.2}$$

$$f\mathcal{L}_{WZ} = \mathrm{i}|Z|^{-2}d\theta^{\dot{\mathsf{a}}i}Z_{ij}d\theta^j_{\dot{\mathsf{a}}} + \int_0^1 dt\Big[2|Z|^{-1}\hat{L}^{\mathsf{a}\dot{\mathsf{b}}}\eta^i_{\mathsf{a}}Z_{ij}\mathsf{L}^j_{Q\dot{\mathsf{b}}} + \mathrm{i}\eta^{\mathsf{a}i}DZ_{ij}\mathsf{L}^j_{S\mathsf{a}}\Big] + h.c.\,, \tag{1.3}$$

where the $2d$ metric $g_{\mu\nu}$, $\mu,\nu = 0,1$ has signature $(-,+)$, $g \equiv -detg_{\mu\nu}$. We use the notation

$$Z_{ij} \equiv \rho^M_{ij} Z^M\,, \qquad |Z|^2 = Z^M Z^M\,, \qquad M = 1,\ldots,6 \tag{1.4}$$

$$f = 2\sqrt{2}\,. \tag{1.5}$$

The 'covariant' derivative DZ^M is defined by

$$DZ^M = dZ^M - \frac{1}{2}\Big(d\theta^{\mathsf{a}}_i(\rho^N\bar\rho^M)^i{}_j\eta^j_{\mathsf{a}} + \eta^{\dot{\mathsf{a}}}_i(\rho^M\bar\rho^N)^i{}_j d\theta^j_{\dot{\mathsf{a}}} + \mathrm{i}\mathsf{L}^{\mathsf{a}\dot{\mathsf{b}}}_P\eta_{\dot{\mathsf{b}}i}(\rho^{MN})^i{}_j\eta^j_{\mathsf{a}}\Big)Z^N\,, \tag{1.6}$$

where the $\rho^M = (\rho^M)^{ij}$, $\bar\rho^M = \rho^M_{ij}$ are 4×4 $SO(6)$ γ matrices in chiral representation, and $\rho^{MN} \equiv \rho^{[M}\bar\rho^{N]}$, $\bar\rho^M = \rho^{M\dagger}$ (see Appendix A). The $\hat L^{\mathsf{a}\dot{\mathsf{b}}}$ are bosonic left invariant Cartan 1-forms while $\mathsf{L}^{\mathsf{a}}_{Qi}$ and $\mathsf{L}^i_{S\mathsf{a}}$ are the fermionic ones. They are given by

$$\hat L^{\mathsf{a}\dot{\mathsf{b}}} = L^{\mathsf{a}\dot{\mathsf{b}}}_P - \frac{1}{2}L^{\mathsf{a}\dot{\mathsf{b}}}_K\,, \qquad L^{\mathsf{a}\dot{\mathsf{b}}}_P = |Z|^{-1}\mathsf{L}^{\mathsf{a}\dot{\mathsf{b}}}_P\,, \qquad L^{\mathsf{a}\dot{\mathsf{b}}}_K = |Z|\mathsf{L}^{\mathsf{a}\dot{\mathsf{b}}}_K\,, \tag{1.7}$$

$$\mathsf{L}^{\mathsf{a}\dot{\mathsf{b}}}_P = dx^{\mathsf{a}\dot{\mathsf{b}}} - \frac{\mathrm{i}}{2}(\theta^{\mathsf{a}}_i d\theta^{\dot{\mathsf{b}}i} + \theta^{\dot{\mathsf{b}}i}d\theta^{\mathsf{a}}_i)\,, \tag{1.8}$$

$$\mathsf{L}^{\mathsf{a}\dot{\mathsf{b}}}_K = \frac{\mathrm{i}}{2}\eta^{\mathsf{a}i}d\eta^{\dot{\mathsf{b}}}_i + \frac{\mathrm{i}}{12}(d\theta^{\mathsf{c}}_i - \frac{\mathrm{i}}{4}\mathsf{L}^{\mathsf{c}\dot{\mathsf{d}}}_P\eta_{\dot{\mathsf{d}}i})(5\eta^j_{\mathsf{c}}\eta^{\mathsf{a}i} - \eta^i_{\mathsf{c}}\eta^{\mathsf{a}j} - \eta^{\mathsf{d}i}\eta^j_{\mathsf{d}}\delta^{\mathsf{a}}_{\mathsf{c}})\eta^{\dot{\mathsf{b}}}_j + h.c.\ \ , \tag{1.9}$$

$$\mathsf{L}^{\dot{\mathsf{a}}i}_Q = d\theta^{\dot{\mathsf{a}}i} - \mathrm{i}\eta^i_{\mathsf{b}}\mathsf{L}^{\mathsf{b}\dot{\mathsf{a}}}_P\,, \qquad \mathsf{L}^{\mathsf{a}}_{Qi} = d\theta^{\mathsf{a}}_i - \mathrm{i}\mathsf{L}^{\mathsf{a}\dot{\mathsf{b}}}_P\eta_{\dot{\mathsf{b}}i}\,, \tag{1.10}$$

$$\mathsf{L}^{\mathsf{a}i}_S = d\eta^{\mathsf{a}i} + \frac{1}{4}(d\theta^{\mathsf{c}}_j - \frac{\mathrm{i}}{3}\mathsf{L}^{\mathsf{c}\dot{\mathsf{d}}}_P\eta_{\dot{\mathsf{d}}j})(\eta^j_{\mathsf{c}}\eta^{\mathsf{a}i} - 3\eta^i_{\mathsf{c}}\eta^{\mathsf{a}j} + \eta^{\mathsf{d}j}\eta^i_{\mathsf{d}}\delta^{\mathsf{a}}_{\mathsf{c}}) + \frac{1}{2}(d\theta^{\dot{\mathsf{c}}i} - \frac{\mathrm{i}}{3}\eta^i_{\mathsf{b}}\mathsf{L}^{\mathsf{b}\dot{\mathsf{c}}}_P)\eta_{\dot{\mathsf{c}}j}\eta^{\mathsf{a}j}\,. \tag{1.11}$$

Their world-sheet projections are defined as usual by $L = d\sigma^\mu L_\mu$. Hermitean conjugation in (1.9) should be supplemented by $\mathsf{a} \leftrightarrow \mathsf{b}$. The $x^{\mathsf{a}\dot{\mathsf{b}}}$ is expressible in terms of the coordinates x^a as

$$x^{\mathsf{a}\dot{\mathsf{b}}} = \frac{1}{\sqrt{2}}(\sigma^a)^{\mathsf{a}\dot{\mathsf{b}}}x^a\,, \tag{1.12}$$

where $(\sigma^a)^{\mathsf{a}\dot{\mathsf{b}}}$ are 2×2 $SO(3,1)$ γ matrices in chiral representation (see Appendix A). Note that in expressions for $\hat L^{\mathsf{a}\dot{\mathsf{b}}}$, $\mathsf{L}^i_{Q\dot{\mathsf{a}}}$, $\mathsf{L}^i_{S\mathsf{a}}$, DZ^M in $\mathcal{L}_{WZ}$ (1.3) the η should be shifted $\eta \to t\eta$. In (1.3) the products of 1-forms, say L^1 and L^2, should read as $\epsilon^{\mu\nu}L^1_\mu L^2_\nu$. The above action corresponds to the "4 + 6" choice of conformally flat coordinates in $AdS_5 \times S^5$ space

$$ds^2 = \frac{1}{Z^2}(dx^a dx^a + dZ^M dZ^M)\,. \tag{1.13}$$

Here and below we set the radius of AdS_5 and S^5 to 1. We finish this discussion with few remarks.

(i) The Lagrangian (1.1) is manifestly invariant with respect to $SL(2,C) \times SU(4)$.

(ii) The dependence on boundary coordinates is through their derivatives dx^a, i.e. the invariance with respect to boundary Poincaré translations is manifest.

(iii) The Lagrangian depends on Poincaré fermionic coordinates θ either through their derivatives $d\theta$ or fermionic 1-forms $\mathsf{L}_P^{\mathsf{a}\dot{\mathsf{b}}}$. Because both of these quantities are invariant with respect to $\mathcal{N}=4$ Poincaré supersymmetries

$$\delta\theta^{\dot{\mathsf{a}}i} = \epsilon^{\dot{\mathsf{a}}i}\,, \qquad \delta\theta_i^{\mathsf{a}} = \epsilon_i^{\mathsf{a}}\,, \qquad \delta x^{\mathsf{a}\dot{\mathsf{b}}} = \frac{\mathrm{i}}{2}(\epsilon_i^{\mathsf{a}}\theta^{\dot{\mathsf{b}}i} + \epsilon^{\dot{\mathsf{b}}i}\theta_i^{\mathsf{a}})\,, \tag{1.14}$$

the Lagrangian is also manifestly invariant with respect to $\mathcal{N}=4$ Poincaré supersymmetry.

(iv) The Lagrangian is manifestly invariant with respect to dilatation symmetries which are realized as follows

$$\delta x^{\mathsf{a}\dot{\mathsf{b}}} = e^{\lambda}x^{\mathsf{a}\dot{\mathsf{b}}}\,, \quad \delta Z^M = e^{\lambda}Z^M\,, \quad \delta\phi = -\lambda\,, \quad \delta\theta = e^{\lambda/2}\theta\,, \quad \delta\eta = e^{-\lambda/2}\eta\,. \tag{1.15}$$

(v) The Lagrangian involves terms of 4th degree in θ and 8th degree in η. Maximal degree of fermionic coordinates appearing in Lagrangian is equal to 12.

(vi) The S gauge fixed action is obtainable by setting $\eta = 0$. In this gauge we have $L_K^{\mathsf{a}\dot{\mathsf{b}}} = 0$, $\mathsf{L}_S^{\mathsf{a}i} = 0$, $DZ^M = dZ^M$. With these relations we get immediately the S gauged action [5, 6, 10] in the form given in [10] (see formulas (C.12), (C.14) in Ref. [10]).

(vii) Light-cone gauge fixed action of [10] is obtainable by setting

$$\theta_i^2 = \theta_{1i} = \theta^{\dot{2}i} = \theta_{\dot{1}}^i = \eta^{2i} = \eta_1^i = \eta_i^{\dot{2}} = \eta_{\dot{1}i} = 0\,, \tag{1.16}$$

$$\theta^{\dot{1}i} = -\theta_{\dot{2}}^i = -\theta^i\,, \qquad \theta_i^1 = \theta_{2i} = -\theta_i\,, \qquad \eta^{1i} = \eta_2^i = \eta^i\,, \qquad \eta_i^{\dot{1}} = -\eta_{\dot{2}i} = \eta_i\,. \tag{1.17}$$

Note that due to index raising and lowering rules (A.5) and Hermitean conjugation rules

$$(\theta_i^{\mathsf{a}})^\dagger = \theta^{\dot{\mathsf{a}}i}\,, \quad (\theta_{\mathsf{a}i})^\dagger = -\theta_{\dot{\mathsf{a}}}^i\,, \quad (\eta^{\mathsf{a}i})^\dagger = \eta_i^{\dot{\mathsf{a}}}\,, \quad (\eta_{\mathsf{a}}^i)^\dagger = -\eta_{\dot{\mathsf{a}}i}\,, \tag{1.18}$$

only one quarter of the relations (1.16),(1.17) are independent. In light-cone gauge all bilinear expressions for fermions with contracted $sl(2)$ indices are equal to zero, e.g.,

$$\theta_i^{\mathsf{a}}\theta_{\mathsf{a}j} = 0\,, \qquad \eta^{i\mathsf{a}}\eta_{\mathsf{a}}^j = 0\,, \qquad \theta_i^{\mathsf{a}}\eta_{\mathsf{a}}^j = 0\,. \tag{1.19}$$

Note that due to this rule the first and the last terms in $\mathcal{L}_{WZ}$ (1.3) are equal to zero. Taking into account that in light-cone gauge $L_K^{2\dot{2}} = L_K^{1\dot{2}} = L_K^{2\dot{1}} = L_S^{2i} = L_{Si}^{\dot{2}} = 0$ and the relation between $sl(2)$ and light-cone notation

$$x^{1\dot{1}} = x^-\,, \qquad x^{2\dot{2}} = -x^+\,, \qquad x^{1\dot{2}} = \bar{x}\,, \qquad x^{2\dot{1}} = x\,, \tag{1.20}$$

(see Appendix A) and making field redefinitions

$$\eta^i \to \sqrt{2}|Z|^{-1}\eta^i, \qquad \eta_i \to \sqrt{2}|Z|^{-1}\eta_i, \qquad\qquad x^a \to -x^a\,, \tag{1.21}$$

we get from (1.2),(1.3) the superstring kappa symmetry light-cone gauge fixed action found in [10]

$$\mathcal{L}_{kin} = -\sqrt{g} g^{\mu\nu} Z^{-2} \Big[\partial_\mu x^+ \partial_\nu x^- + \partial_\mu x \partial_\nu \bar{x} + \frac{1}{2} D_\mu Z^M D_\nu Z^M \Big]$$

$$- \frac{\mathrm{i}}{2} \sqrt{g} g^{\mu\nu} Z^{-2} \partial_\mu x^+ \Big[\theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + \mathrm{i} Z^{-2} \partial_\nu x^+ (\eta^2)^2 \Big] \,, \qquad (1.22)$$

$$\mathcal{L}_{WZ} = \epsilon^{\mu\nu} |Z|^{-3} \partial_\mu x^+ \eta^i \rho^M_{ij} Z^M (\partial_\nu \theta^j - \mathrm{i}\sqrt{2} |Z|^{-1} \eta^j \partial_\nu x) + h.c. \,, \qquad (1.23)$$

where in light-cone gauge (1.16),(1.17) the covariant derivative (1.6) reduces to

$$DZ^M = dZ^M - 2\mathrm{i} \eta_i (R^M)^i{}_j \eta^j Z^{-2} dx^+ \,, \qquad R^M = -\frac{1}{2} \rho^{MN} Z^N \,. \qquad (1.24)$$

(viii) By applying phase space GGRT [14] approach based on fixing diffeomorphisms by $x^+ = \tau$, $\mathcal{P}^+ = p^+ = const$, where $\mathcal{P}^+$ is a canonical momentum, the Lagrangian can be put into the form found in [11]

$$\begin{aligned}
\mathcal{L} &= \mathcal{P}_\perp \dot{x}_\perp + \mathcal{P}_M \dot{Z}^M + \frac{\mathrm{i}}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) \\
&- \frac{1}{2p^+} \Big[\mathcal{P}_\perp^2 + \mathcal{P}_M^2 + |Z|^{-4} (\acute{x}_\perp^2 + \acute{Z}_M^2) + |Z|^{-2} (p^{+2} (\eta^2)^2 + 4\mathrm{i} p^+ \eta R^M \eta \mathcal{P}_M) \Big] \\
&+ \Big[\, |Z|^{-3} \eta^i \rho^M_{ij} Z^M (\acute{\theta}^j - \mathrm{i}\sqrt{2} |Z|^{-1} \eta^j \acute{x}) + h.c. \Big] \qquad (1.25)
\end{aligned}$$

where $x^\perp = (x, \bar{x})$, $\dot{x} = \partial_\tau x$, $\acute{x} \equiv \partial_\sigma x$. $\mathcal{P}^\perp = (\mathcal{P}, \bar{\mathcal{P}})$ and $\mathcal{P}^M$ are canonical momenta for $(\bar{x}, x)$ and Z^M respectively. The interesting feature of this Lagrangian is that the squares of the AdS_5 and S^5 momenta enter exactly as in the flat space case. This implies that solution to phase space light-cone gauge equations of motion of *bosonic particle* in $AdS_5 \times S^5$ takes the same form as in flat space

$$\mathcal{P}^\perp = \mathcal{P}_0^\perp \,, \quad x^\perp = x_0^\perp + \frac{\mathcal{P}_0^\perp}{p^+} \tau \,, \qquad \mathcal{P}^M = \mathcal{P}_0^M \,, \quad Z^M = Z_0^M + \frac{\mathcal{P}_0^M}{p^+} \tau \,, \qquad (1.26)$$

where the variables carrying subscript 0 are initial data. An intersting feature of this solution is that as compared to the one discussed in [12] our solution does not involve trigonometric functions. This simplification is due to (i) choice of conformally flat parametrization of $AdS_5 \times S^5$ background; (ii) fixing diffeomorphisms by light-cone gauge [11]. Note that in the case of bosonic particle such simple solution can be also reached by choosing covariant gauge on world line vielbein $e = |Z|^{-2}$ which is similar to the conformal gauge suggested for the string in [15]. Taking this into account we conclude that at least in particle approximation the conformally flat coordinates (1.13) are most suitable for analysis of AdS string dynamics.

In the rest of the paper we expalain the procedure of derivation of the manifestly $SL(2,C) \times SU(4)$ action from the original superstring action [2] which has manifest $SO(4,1) \times SO(5)$ invariance.

2 Superstring action in $so(4,1) \oplus so(5)$ and $sl(2,C) \oplus su(4)$ bases

Superstring Lagrangian in $AdS_5 \times S^5$ [2] in $so(4,1) \oplus so(5)$ basis of $psu(2,2|4)$ superalgebra has the same structure as the flat space GS action [16, 17]

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{WZ}\,, \qquad \mathcal{L}_{WZ} = d^{-1}(\mathrm{i}\mathcal{H})\ , \tag{2.1}$$

where the kinetic term of the $AdS_5 \times S^5$ GS action $\mathcal{L}_{kin}$ and the WZ three form $\mathcal{H}$ are expressed in terms of the Cartan 1-forms and have the following form in the $so(4,1) \oplus so(5)$ basis [2]

$$\mathcal{L}_{kin} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}(\hat{L}^A_\mu \hat{L}^A_\nu + L^{\mathrm{A}'}_\mu L^{\mathrm{A}'}_\nu)\ , \tag{2.2}$$

$$\mathcal{H} = s^{IJ}\hat{L}^A \bar{L}^I \gamma^A L^J + \mathrm{i}s^{IJ} L^{\mathrm{A}'} \bar{L}^J \gamma^{\mathrm{A}'} L^J\ , \tag{2.3}$$

where $s^{IJ} = diag(1,-1)$, $I, J = 1,2$ and $\bar{L}^I \equiv L^I CC'$. The γ^A and $\gamma^{\mathrm{A}'}$ are $SO(4,1)$ and $SO(5)$ Dirac γ matrices respectively while C and C' are appropriate charge conjugation matrices. The left invariant bosonic Cartan 1-forms L^A, $L^{\mathrm{A}'}$ and fermionic ones L^I are defined by decomposition

$$G^{-1}dG = (G^{-1}dG)_{bos} + L^{I\alpha i}Q_{I\alpha i}\ , \tag{2.4}$$

where G is a coset representative of $PSU(2,2|4)$ group and the restriction to the bosonic part is

$$(G^{-1}dG)_{bos} = \hat{L}^A \hat{P}^A + \frac{1}{2}\hat{L}^{AB}\hat{J}^{AB} + L^{\mathrm{A}'}P^{\mathrm{A}'} + \frac{1}{2}L^{\mathrm{A}'\mathrm{B}'}J^{\mathrm{A}'\mathrm{B}'}\ . \tag{2.5}$$

The generators $\hat{P}^A$, $\hat{J}^{AB}$, and $P^{\mathrm{A}'}$, $J^{\mathrm{A}'\mathrm{B}'}$ are translations and rotations generators for AdS_5 and S^5 respectively, while Q_I are fermionic generators (see Appendix B). The remarkable property of $so(4,1) \oplus so(5)$ basis is that it is this basis that allows one to present the $AdS_5 \times S^5$ GS action and κ symmetry transformations in the form similar to the one in the flat space (see [2]).

Our present goal is to rewrite the action in the $sl(2,C) \oplus su(4)$ basis. To do that we shall use the conformal algebra and $su(4)$ notation. We introduce the Poincaré translations P^a, the conformal boosts K^a, the dilatation D, $so(3,1)$ Lorentz algebra generators J^{ab} and $su(4)$ generators $J^i{}_j$ by relations

$$P^a = \hat{P}^a + \hat{J}^{4a}\ , \qquad K^a = \frac{1}{2}(-\hat{P}^a + \hat{J}^{4a})\ , \qquad D = -\hat{P}^4\,, \qquad J^{ab} = \hat{J}^{ab}\ , \tag{2.6}$$

$$J^i{}_j \equiv -\frac{\mathrm{i}}{2}(\gamma^{\mathrm{A}'})^i{}_j P^{\mathrm{A}'} + \frac{1}{4}(\gamma^{\mathrm{A}'\mathrm{B}'})^i{}_j J^{\mathrm{A}'\mathrm{B}'}\ . \tag{2.7}$$

In conformal algebra notation we have the following decomposition

$$(G^{-1}dG)_{bos} = L^a_{\scriptscriptstyle P} P^a + L^a_{\scriptscriptstyle K} K^a + L_{\scriptscriptstyle D} D + \frac{1}{2}L^{ab}J^{ab} + L^i{}_j J^j{}_i\ . \tag{2.8}$$

Comparing this with (2.5) and using (2.6), (2.7) we get interrelation of Cartan 1-forms in $so(4,1)\oplus so(5)$ and conformal bases

$$\hat{L}^a = L_P^a - \frac{1}{2}L_K^a\,, \qquad \hat{L}^{4a} = L_P^a + \frac{1}{2}L_K^a\,, \qquad \hat{L}^4 = -L_D\,, \qquad L^{\mathrm{A}'} = -\frac{\mathrm{i}}{2}(\gamma^{\mathrm{A}'})^i{}_j L^j{}_i\,, \tag{2.9}$$

$$L^i{}_j = \frac{\mathrm{i}}{2}(\gamma^{\mathrm{A}'})^i{}_j L^{\mathrm{A}'} - \frac{1}{4}(\gamma^{\mathrm{A}'\mathrm{B}'})^i{}_j L^{\mathrm{A}'\mathrm{B}'}\,. \tag{2.10}$$

Using these representations for $\hat{L}^a$, L_D, $L^{\mathrm{A}'}$ allows us transform kinetic term to conformal algebra notation. In what follows we prefer use the $sl(2,C)$ notation instead of the $so(3,1)$ one. To this end we introduce generators in $sl(2)$ basis

$$P_{\mathsf{a}\dot{\mathsf{b}}} = \frac{1}{\sqrt{2}}(\sigma^a)_{\mathsf{a}\dot{\mathsf{b}}}P^a\,, \qquad K_{\mathsf{a}\dot{\mathsf{b}}} = \frac{1}{\sqrt{2}}(\sigma^a)_{\mathsf{a}\dot{\mathsf{b}}}K^a\,, \tag{2.11}$$

$$J_{\mathsf{a}\mathsf{b}} = \frac{1}{2}(\sigma^{ab})_{\mathsf{a}\mathsf{b}}J^{ab}\,, \qquad J_{\dot{\mathsf{a}}\dot{\mathsf{b}}} = -\frac{1}{2}(\bar{\sigma}^{ab})_{\dot{\mathsf{a}}\dot{\mathsf{b}}}J^{ab}\,, \tag{2.12}$$

and corresponding bosonic $sl(2)$ Cartan 1-forms

$$L_P^{\mathsf{a}\dot{\mathsf{b}}} = \frac{1}{\sqrt{2}}(\sigma^a)^{\mathsf{a}\dot{\mathsf{b}}}L_P^a\,, \qquad L_K^{\mathsf{a}\dot{\mathsf{b}}} = \frac{1}{\sqrt{2}}(\sigma^a)^{\mathsf{a}\dot{\mathsf{b}}}L_K^a\,, \tag{2.13}$$

$$L^{\mathsf{a}\mathsf{b}} = \frac{1}{2}(\sigma^{ab})^{\mathsf{a}\mathsf{b}}L^{ab}\,, \qquad L^{\dot{\mathsf{a}}\dot{\mathsf{b}}} = -\frac{1}{2}(\bar{\sigma}^{ab})^{\dot{\mathsf{a}}\dot{\mathsf{b}}}L^{ab}\,. \tag{2.14}$$

Transformation of supergenerators and fermionic Cartan 1-forms to $sl(2)\oplus su(4)$ basis may be found in Appendix B. In the $sl(2)\oplus su(4)$ notation Cartan 1-form are obtainable from the decompositon

$$\begin{aligned} G^{-1}dG \;&=\; L_P^{\mathsf{a}\dot{\mathsf{b}}}P_{\mathsf{a}\dot{\mathsf{b}}} + L_K^{\mathsf{a}\dot{\mathsf{b}}}K_{\mathsf{a}\dot{\mathsf{b}}} + L_D D + \frac{1}{4}(L^{\mathsf{a}\mathsf{b}}J_{\mathsf{a}\mathsf{b}} + L^{\dot{\mathsf{a}}\dot{\mathsf{b}}}J_{\dot{\mathsf{a}}\dot{\mathsf{b}}}) + L^i{}_j J^j{}_i \\ &+\; L^{\mathsf{a}}_{Qi}Q^i_{\mathsf{a}} - L_Q^{\dot{\mathsf{a}}i}Q_{\dot{\mathsf{a}}i} + L_S^{\mathsf{a}i}S_{\mathsf{a}i} - L^{\dot{\mathsf{a}}}_{Si}S^i_{\dot{\mathsf{a}}}\,, \end{aligned} \tag{2.15}$$

where Q and S are Poincaré and conformal supercharges respectively, while L_Q and L_S are appropriate fermionic Cartan 1-forms (for details see Appendix B). The kinetic term and WZ three form take then the form

$$\mathcal{L}_{kin} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}\Big(\hat{L}_\mu^{\mathsf{a}\dot{\mathsf{b}}}\hat{L}_{\nu\mathsf{a}\dot{\mathsf{b}}} + L_{D\mu}L_{D\nu} + L_\mu^{\mathrm{A}'}L_\nu^{\mathrm{A}'}\Big)\,, \tag{2.16}$$

$$\mathcal{H} = \mathcal{H}^q_{AdS_5} + \mathcal{H}^q_{S^5} - h.c.\,, \tag{2.17}$$

where AdS_5 and S^5 contributions in WZ tree form are given by

$$f\mathcal{H}^q_{AdS_5} \;=\; 2\mathrm{i}\hat{L}^{\mathsf{a}\dot{\mathsf{b}}}L^i_{S\mathsf{a}}C'_{ij}L^j_{Q\dot{\mathsf{b}}} + L_D(\frac{1}{2}L_S^{\mathsf{a}i}C'_{ij}L^j_{S\mathsf{a}} + L_Q^{\dot{\mathsf{a}}i}C'_{ij}L^j_{Q\dot{\mathsf{a}}})\,, \tag{2.18}$$

$$f\mathcal{H}^q_{S^5} \;=\; L_S^{\mathsf{a}i}(C'L)_{ij}L^j_{S\mathsf{a}} - 2L_Q^{\dot{\mathsf{a}}i}(C'L)_{ij}L^j_{Q\dot{\mathsf{a}}}\,. \tag{2.19}$$

We use the notation

$$(C'L)_{ij} \equiv C'_{ik} L^k{}_j \,, \tag{2.20}$$

where C'_{ij} is a charge conjugation matrix of $SO(5)$ Dirac γ matrices, $\gamma^T_{\mathrm{A}'} = C'\gamma_{\mathrm{A}'}C'^{-1}$. Note that $L^{\mathrm{A}'}$ is expressible in terms of $L^i{}_j$ due to last relation in (2.9). The formulas (2.15)-(2.19) give description of $AdS_5 \times S^5$ superstring action in conformal algebra notation. They are conformal counterpart of $so(4,1) \oplus so(5)$ representation given in (2.2)-(2.4).

3 $psu(2,2|4)$ superalgebra and Cartan forms in $sl(2) \oplus su(4)$ basis

To find superstring action in $sl(2) \oplus su(4)$ basis we need commutation relations and Cartan 1-forms in this basis. In this basis bosonic part of $psu(2,2|4)$ superalgebra consists of Poincaré translations $P_{\mathsf{a}\dot{\mathsf{b}}}$, conformal boosts $K_{\mathsf{a}\dot{\mathsf{b}}}$, dilatation D, Lorentz algebra generators J_{ab}, $J_{\dot{\mathsf{a}}\dot{\mathsf{b}}}$ and $su(4)$ algebra generators $J^i{}_j$. We adopt the following commutation relations between bosonic generators

$$[P_{\mathsf{a}\dot{\mathsf{b}}}, K_{\mathsf{c}\dot{\mathsf{d}}}] = \epsilon_{\mathsf{ac}}\epsilon_{\dot{\mathsf{b}}\dot{\mathsf{d}}} D + \frac{1}{2}(\epsilon_{\mathsf{ac}} J_{\dot{\mathsf{b}}\dot{\mathsf{d}}} + \epsilon_{\dot{\mathsf{b}}\dot{\mathsf{d}}} J_{\mathsf{ac}}) \,, \tag{3.1}$$

$$[J_{\mathsf{ab}}, J_{\mathsf{cd}}] = \epsilon_{\mathsf{bc}} J_{\mathsf{ad}} + 3 \text{ terms} \quad , \qquad [J^i{}_j, J^k{}_n] = \delta^k_j J^i{}_n - \delta^i_n J^k{}_j \,, \tag{3.2}$$

$$[D, P_{\mathsf{a}\dot{\mathsf{b}}}] = -P_{\mathsf{a}\dot{\mathsf{b}}} \,, \qquad [D, K_{\mathsf{a}\dot{\mathsf{b}}}] = K_{\mathsf{a}\dot{\mathsf{b}}} \,, \tag{3.3}$$

$$[P_{\mathsf{a}\dot{\mathsf{b}}}, J_{\mathsf{cd}}] = \epsilon_{\mathsf{ac}} P_{\mathsf{d}\dot{\mathsf{b}}} + \epsilon_{\mathsf{ad}} P_{\mathsf{c}\dot{\mathsf{b}}} \,, \qquad [K_{\mathsf{a}\dot{\mathsf{b}}}, J_{\mathsf{cd}}] = \epsilon_{\mathsf{ac}} K_{\mathsf{d}\dot{\mathsf{b}}} + \epsilon_{\mathsf{ad}} K_{\mathsf{c}\dot{\mathsf{b}}} \,. \tag{3.4}$$

Note that because J_{ab} is symmetric in a,b the remaining three terms in the first commutator in (3.2) are obtainable by symmetrization in a,b and c,d. Fermionic part of $psu(2,2|4)$ superalgebra consists of Poincaré supergenerators Q^i_{a}, $Q_{\dot{\mathsf{a}}i}$ and conformal supercharges $S_{\mathsf{a}i}$, $S^i_{\dot{\mathsf{a}}}$. Commutation relations between bosonic and fermionic generators take the form

$$[J_{\mathsf{ab}}, Q^i_{\mathsf{c}}] = \epsilon_{\mathsf{bc}} Q^i_{\mathsf{a}} + \epsilon_{\mathsf{ac}} Q^i_{\mathsf{b}} \,, \qquad [J^i{}_j, Q^k_{\mathsf{a}}] = \delta^k_j Q^i_{\mathsf{a}} - \frac{1}{4}\delta^i_j Q^k_{\mathsf{a}} \,, \tag{3.5}$$

$$[J_{\mathsf{ab}}, S_{\mathsf{c}i}] = \epsilon_{\mathsf{bc}} S_{\mathsf{a}i} + \epsilon_{\mathsf{ac}} S_{\mathsf{b}i} \,, \qquad [J^i{}_j, S_{\mathsf{a}k}] = -\delta^i_k S_{\mathsf{a}j} + \frac{1}{4}\delta^i_j S_{\mathsf{a}k} \,, \tag{3.6}$$

$$[D, Q^i_{\mathsf{a}}] = -\frac{1}{2} Q^i_{\mathsf{a}} \,, \quad [D, S_{\mathsf{a}i}] = \frac{1}{2} S_{\mathsf{a}i} \,, \tag{3.7}$$

$$[S_{\mathsf{a}i}, P_{\mathsf{b}\dot{\mathsf{c}}}] = -\mathrm{i}\epsilon_{\mathsf{ab}} Q_{\dot{\mathsf{c}}i} \,, \qquad [Q^i_{\mathsf{a}}, K_{\mathsf{b}\dot{\mathsf{c}}}] = \mathrm{i}\epsilon_{\mathsf{ab}} S^i_{\dot{\mathsf{c}}} \,. \tag{3.8}$$

Anticommutation relations between fermionic generators take the form

$$\{Q^i_{\mathsf{a}}, Q_{\dot{\mathsf{b}}j}\} = \mathrm{i}P_{\mathsf{a}\dot{\mathsf{b}}}\delta^i_j\,, \qquad \{S_{\mathsf{a}i}, S^j_{\dot{\mathsf{b}}}\} = -\mathrm{i}K_{\mathsf{a}\dot{\mathsf{b}}}\delta^j_i\,, \tag{3.9}$$

$$\{Q^i_{\mathsf{a}}, S_{\mathsf{b}j}\} = (\frac{1}{2}\epsilon_{\mathsf{ab}}D + \frac{1}{2}J_{\mathsf{ab}})\delta^i_j + \epsilon_{\mathsf{ab}}J^i{}_j\,. \tag{3.10}$$

Remaining nontrivial (anti)commutators can be obtained from the above ones via Hermitean conjugation rule which take the following form

$$P^\dagger_{\mathsf{a}\dot{\mathsf{b}}} = -P_{\mathsf{b}\dot{\mathsf{a}}}\,, \qquad K^\dagger_{\mathsf{a}\dot{\mathsf{b}}} = -K_{\mathsf{b}\dot{\mathsf{a}}}\,, \quad J^\dagger_{\mathsf{ab}} = J_{\dot{\mathsf{a}}\dot{\mathsf{b}}}\,, \quad (J^i{}_j)^\dagger = J^j{}_i\,, \quad D^\dagger = -D\,. \tag{3.11}$$

$$(Q^i_{\mathsf{a}})^\dagger = -Q_{\dot{\mathsf{a}}i}\,, \quad (Q^{\mathsf{a}i})^\dagger = Q^{\dot{\mathsf{a}}}_i\,, \quad (S_{\mathsf{a}i})^\dagger = -S^i_{\dot{\mathsf{a}}}\,, \quad (S^{\mathsf{a}}_i)^\dagger = S^{\dot{\mathsf{a}}i}\,. \tag{3.12}$$

The left invariant Cartan 1-forms in $sl(2) \oplus su(4)$ basis are defined by relation (2.15). They satisfy the Maurer-Cartan equations implied by the structure of $psu(2,2|4)$ superalgebra

$$\begin{aligned}
dL^{\mathsf{a}\dot{\mathsf{b}}}_P &= L_D \wedge L^{\mathsf{a}\dot{\mathsf{b}}}_P - \frac{1}{2}(L^{\mathsf{a}}{}_{\mathsf{c}} \wedge L^{\mathsf{c}\dot{\mathsf{b}}}_P + L^{\dot{\mathsf{b}}}{}_{\dot{\mathsf{c}}} \wedge L^{\mathsf{a}\dot{\mathsf{c}}}_P) - \mathrm{i}L^{\mathsf{a}}_{Qi} \wedge L^{\dot{\mathsf{b}}i}_Q\,, && (3.13)\\
dL^{\mathsf{a}\dot{\mathsf{b}}}_K &= -L_D \wedge L^{\mathsf{a}\dot{\mathsf{b}}}_K - \frac{1}{2}(L^{\mathsf{a}}{}_{\mathsf{c}} \wedge L^{\mathsf{c}\dot{\mathsf{b}}}_K + L^{\dot{\mathsf{b}}}{}_{\dot{\mathsf{c}}} \wedge L^{\mathsf{a}\dot{\mathsf{c}}}_K) + \mathrm{i}L^{\mathsf{a}i}_S \wedge L^{\dot{\mathsf{b}}}_{Si}\,, && (3.14)\\
dL_D &= -L^{\mathsf{a}\dot{\mathsf{b}}}_P \wedge L_{K\mathsf{a}\dot{\mathsf{b}}} - \frac{1}{2}L^{\mathsf{a}}_{Qi} \wedge L^i_{S\mathsf{a}} - \frac{1}{2}L^{\dot{\mathsf{a}}i}_Q \wedge L_{S\dot{\mathsf{a}}i}\,, && (3.15)\\
dL^i{}_j &= L^i{}_n \wedge L^n{}_j - L^{\mathsf{a}}_{Qj} \wedge L^i_{S\mathsf{a}} + L^{\dot{\mathsf{a}}i}_Q \wedge L_{S\dot{\mathsf{a}}j} - \mathrm{Tr\ in}\ i,j\,, && (3.16)\\
dL^{\dot{\mathsf{a}}i}_Q &= \frac{1}{2}L_D \wedge L^{\dot{\mathsf{a}}i}_Q - \frac{1}{2}L^{\dot{\mathsf{a}}}{}_{\dot{\mathsf{b}}} \wedge L^{\dot{\mathsf{b}}i}_Q + L^i{}_j \wedge L^{\dot{\mathsf{a}}j}_Q + \mathrm{i}L^{\mathsf{b}\dot{\mathsf{a}}}_P \wedge L^i_{S\mathsf{b}}\,, && (3.17)\\
dL^{\mathsf{a}}_{Qi} &= \frac{1}{2}L_D \wedge L^{\mathsf{a}}_{Qi} - \frac{1}{2}L^{\mathsf{a}}{}_{\mathsf{b}} \wedge L^{\mathsf{b}}_{Qi} - L^j{}_i \wedge L^{\mathsf{a}}_{Qj} + \mathrm{i}L^{\mathsf{a}\dot{\mathsf{b}}}_P \wedge L_{S\dot{\mathsf{b}}i}\,, && (3.18)\\
dL^{\mathsf{a}i}_S &= -\frac{1}{2}L_D \wedge L^{\mathsf{a}i}_S - \frac{1}{2}L^{\mathsf{a}}{}_{\mathsf{b}} \wedge L^{\mathsf{b}i}_S + L^i{}_j \wedge L^{\mathsf{a}j}_S - \mathrm{i}L^{\mathsf{a}\dot{\mathsf{b}}}_K \wedge L^i_{Q\dot{\mathsf{b}}}\,, && (3.19)\\
dL^{\dot{\mathsf{a}}}_{Si} &= -\frac{1}{2}L_D \wedge L^{\dot{\mathsf{a}}}_{Si} - \frac{1}{2}L^{\dot{\mathsf{a}}}{}_{\dot{\mathsf{b}}} \wedge L^{\dot{\mathsf{b}}}_{Si} - L^j{}_i \wedge L^{\dot{\mathsf{a}}}_{Sj} - \mathrm{i}L^{\mathsf{b}\dot{\mathsf{a}}}_K \wedge L_{Q\mathsf{b}i}\,, && (3.20)
\end{aligned}$$

where the Tr in (3.16) respects the condition $dL^i{}_i = 0$. There are Maurer-Cartan equations for dL^{ab} and $dL^{\dot{\mathsf{a}}\dot{\mathsf{b}}}$ but we do not need them in what follows. Hermitean conjugation rules for Cartan 1-forms take the form[1]

$$L^{\mathsf{a}\dot{\mathsf{b}}*}_{P,K} = L^{\mathsf{b}\dot{\mathsf{a}}}_{P,K}\,, \qquad L^*_D = L_D\,, \qquad L^{i*}_j = -L^j{}_i\,, \qquad L^{\mathsf{ab}*} = -L^{\dot{\mathsf{a}}\dot{\mathsf{b}}}\,, \tag{3.21}$$

$$(L^{\mathsf{a}}_{Qi})^\dagger = L^{\dot{\mathsf{a}}i}_Q\,, \quad (L_{Q\mathsf{a}i})^\dagger = -L^i_{Q\dot{\mathsf{a}}}\,, \quad (L^{\mathsf{a}i}_S)^\dagger = L^{\dot{\mathsf{a}}}_{Si}\,, \quad (L^i_{S\mathsf{a}})^\dagger = -L_{S\dot{\mathsf{a}}i}\,. \tag{3.22}$$

[1]For fermionic coordinates we assume the convention $(\theta_1\theta_2)^\dagger = \theta_2^\dagger\theta_1^\dagger$, $\theta_1\theta_2 = -\theta_2\theta_1$, while for fermionic Cartan 1-form we adopt $(L_1 \wedge L_2)^\dagger = -L_2^\dagger \wedge L_1^\dagger$, $L_1 \wedge L_2 = L_2 \wedge L_1$.

The above relations are valid for arbitrary parametrization of supercoset space. To represent Cartan 1-forms in terms of bosonic and fermionic superstring coordinate fields we fix the coset representative to be

$$G = g_{x,\theta}\ g_\eta\ g_y\ g_\phi\ , \tag{3.23}$$

where

$$g_{x,\theta} = \exp(x^{\mathsf{a}\dot{\mathsf{b}}}P_{\mathsf{a}\dot{\mathsf{b}}} + \theta_i^{\mathsf{a}}Q^i_{\mathsf{a}} - \theta^{\dot{\mathsf{a}}i}Q_{\dot{\mathsf{a}}i})\ , \qquad g_\eta = \exp(\eta^{\mathsf{a}i}S_{\mathsf{a}i} - \eta_i^{\dot{\mathsf{a}}}S^i_{\dot{\mathsf{a}}})\ , \tag{3.24}$$

and g_ϕ and g_y depend on AdS_5 radial coordinate ϕ and S^5 coordinates $y^{\mathrm{A}'}$ respectively[2]

$$g_y = \exp(y^i{}_j J^j{}_i)\,, \qquad g_\phi = \exp(\phi D)\,. \tag{3.25}$$

The $x^{\mathsf{a}\dot{\mathsf{b}}}$ and $y^i{}_j$ are expressible in terms of the coordinates along boundary directions x^a and S^5 coordinates as follows

$$x^{\mathsf{a}\dot{\mathsf{b}}} = \frac{1}{\sqrt{2}}(\sigma^a)^{\mathsf{a}\dot{\mathsf{b}}}x^a\,, \qquad y^i{}_j = \frac{\mathrm{i}}{2}y^{\mathrm{A}'}(\gamma^{\mathrm{A}'})^i{}_j\,, \qquad x^{\mathsf{a}\dot{\mathsf{b}}*} = x^{\mathsf{b}\dot{\mathsf{a}}}\,, \quad y^{i*}_j = -y^j{}_i\,. \tag{3.26}$$

From these definitions it follows that $x^{\mathsf{a}\dot{\mathsf{b}}}$ and fermionic coordinates θ, η transform in appropriate representations of $sl(2)\oplus su(4)$ algebra. Plugging G (3.23) in (2.15) and after relative straightforward calculation we get the Cartan 1-forms

$$L_P^{\mathsf{a}\dot{\mathsf{b}}} = e^{\phi}\mathsf{L}_P^{\mathsf{a}\dot{\mathsf{b}}}\,, \qquad L_K^{\mathsf{a}\dot{\mathsf{b}}} = e^{-\phi}\mathsf{L}_K^{\mathsf{a}\dot{\mathsf{b}}}\,, \tag{3.27}$$

$$L_D = d\phi + \frac{1}{2}(\eta^{\mathsf{a}i}d\theta_{\mathsf{a}i} + \eta_i^{\dot{\mathsf{a}}}d\theta^i_{\dot{\mathsf{a}}})\,, \tag{3.28}$$

$$L^i{}_j = (dUU^{-1})^i{}_j + \tilde{d\theta}{}^{\mathsf{a}}_j\tilde{\eta}^i_{\mathsf{a}} - \tilde{\eta}^{\dot{\mathsf{a}}}_j\tilde{d\theta}{}^i_{\dot{\mathsf{a}}} - \mathrm{i}\mathsf{L}_P^{\mathsf{a}\dot{\mathsf{b}}}\tilde{\eta}_{\dot{\mathsf{b}}j}\tilde{\eta}^i_{\mathsf{a}} - \text{ Tr in } i,j\ \ , \tag{3.29}$$

$$L_Q^{\dot{\mathsf{a}}i} = e^{\phi/2}U^i{}_j\mathsf{L}_Q^{\dot{\mathsf{a}}j}\,, \qquad L^{\mathsf{a}}_{Qi} = \mathsf{L}^{\mathsf{a}}_{Qj}(U^{-1})^j{}_i e^{\phi/2}\,, \tag{3.30}$$

$$L_S^{\dot{\mathsf{a}}i} = e^{-\phi/2}U^i{}_j\mathsf{L}_S^{\dot{\mathsf{a}}j}\,, \qquad L^{\mathsf{a}}_{Si} = \mathsf{L}^{\mathsf{a}}_{Sj}(U^{-1})^j{}_i e^{-\phi/2}\,, \tag{3.31}$$

where the Tr in (3.29) respects the condition $L^i{}_i = 0$ and we use the notation

$$\mathsf{L}_P^{\mathsf{a}\dot{\mathsf{b}}} = dx^{\mathsf{a}\dot{\mathsf{b}}} - \frac{\mathrm{i}}{2}(\theta_i^{\mathsf{a}}d\theta^{\dot{\mathsf{b}}i} + \theta^{\dot{\mathsf{b}}i}d\theta_i^{\mathsf{a}})\,, \tag{3.32}$$

$$\mathsf{L}_K^{\mathsf{a}\dot{\mathsf{b}}} = \frac{\mathrm{i}}{2}\eta^{\mathsf{a}i}d\eta_i^{\dot{\mathsf{b}}} + \frac{\mathrm{i}}{12}(d\theta_i^{\mathsf{c}} - \frac{\mathrm{i}}{4}\mathsf{L}_P^{\mathsf{c}\dot{\mathsf{d}}}\eta_{\dot{\mathsf{d}}i})(5\eta_{\mathsf{c}}^j\eta^{\mathsf{a}i} - \eta_{\mathsf{c}}^i\eta^{\mathsf{a}j} - \eta^{\mathsf{d}i}\eta_{\mathsf{d}}^j\delta_{\mathsf{c}}^{\mathsf{a}})\eta_j^{\dot{\mathsf{b}}} + h.c.\,, \tag{3.33}$$

[2]Splitting fermionic coordinates in θ and η was introduced in [18, 19] in the study of light-cone gauge dynamics of superparticle in $AdS_5\times S^5$. Light-cone gauge superstring action in $AdS_5\times S^5$ written in terms of these coordinates was found in [10]. In the context of brane dynamics these coordinates were discussed in [20].

$$\mathsf{L}_{Q}^{\dot{\mathsf{a}}i} = d\theta^{\dot{\mathsf{a}}i} - \mathrm{i}\eta_{\mathsf{b}}^{i}\mathsf{L}_{P}^{\mathsf{b}\dot{\mathsf{a}}}\,, \qquad \mathsf{L}_{Qi}^{\mathsf{a}} = d\theta_{i}^{\mathsf{a}} - \mathrm{i}\mathsf{L}_{P}^{\mathsf{a}\dot{\mathsf{b}}}\eta_{\dot{\mathsf{b}}i}\,, \tag{3.34}$$

$$\begin{aligned}\mathsf{L}_{S}^{\mathsf{a}i} &= d\eta^{\mathsf{a}i} + \frac{1}{4}(d\theta_{j}^{\mathsf{c}} - \frac{\mathrm{i}}{3}\mathsf{L}_{P}^{\mathsf{c}\dot{\mathsf{d}}}\eta_{\dot{\mathsf{d}}j})(\eta_{\mathsf{c}}^{j}\eta^{\mathsf{a}i} - 3\eta_{\mathsf{c}}^{i}\eta^{\mathsf{a}j} + \eta^{\mathsf{d}j}\eta_{\mathsf{d}}^{i}\delta_{\mathsf{c}}^{\mathsf{a}}) \\ &+ \frac{1}{2}(d\theta^{\dot{\mathsf{c}}i} - \frac{\mathrm{i}}{3}\eta_{\mathsf{b}}^{i}\mathsf{L}_{P}^{\mathsf{b}\dot{\mathsf{c}}})\eta_{\dot{\mathsf{c}}j}\eta^{\mathsf{a}j}\,. \end{aligned} \tag{3.35}$$

The $\tilde{\theta}$ and $\tilde{d\theta}$ are defined by

$$\tilde{\theta}^{\dot{\mathsf{a}}i} \equiv U^{i}{}_{j}\theta^{\dot{\mathsf{a}}j}\,, \qquad \tilde{\theta}_{\mathsf{a}i} \equiv \theta_{\mathsf{a}j}(U^{-1})^{j}{}_{i}\,, \tag{3.36}$$

$$\tilde{d\theta}^{\dot{\mathsf{a}}i} \equiv U^{i}{}_{j}d\theta^{\dot{\mathsf{a}}j}\,, \qquad \tilde{d\theta}_{\mathsf{a}i} \equiv d\theta_{\mathsf{a}j}(U^{-1})^{j}{}_{i}\,, \tag{3.37}$$

and similar ones for η. Hermitean conjugation in (3.33) should be supplemented by $\mathsf{a} \leftrightarrow \mathsf{b}$. The matrix $U \in SU(4)$ is defined in terms of the S^5 coordinates $y^{i}{}_{j}$ (or $y^{\mathrm{A}'}$, see (3.26)) by $U^{i}{}_{j} = (e^{y})^{i}{}_{j}$. It can be written explicitly as

$$U = \cos\frac{|y|}{2} + \mathrm{i}\gamma^{\mathrm{A}'}n^{\mathrm{A}'}\sin\frac{|y|}{2}\,, \qquad |y| \equiv \sqrt{y^{\mathrm{A}'}y^{\mathrm{A}'}}\,, \quad n^{\mathrm{A}'} \equiv \frac{y^{\mathrm{A}'}}{|y|}\,. \tag{3.38}$$

Because fermionic coordinates defined by (3.23) trasform covariantly the 1-forms defined in (3.34),(3.35) also transform covariantly under $SU(4)$. On the other hand because the matrix $U^{i}{}_{j}$ does not transform covariantly under $SU(4)$ group the Cartan 1-forms defined by (3.30),(3.31) also do not transform covariantly under this group. Below we will demonstrate that superstring action can be expressed entirely in terms of η and 1-forms given by (3.34),(3.35) and this explains why the superstring action can be put into manifestly $SU(4)$ covariant form.

4 Manifestly $SL(2,C) \times SU(4)$ invariant form of superstring action

Plugging the expressions for Cartan 1-froms (3.27)-(3.31) into expressions for kinetic and WZ terms (2.16),(2.17) we can represent the superstring Lagrangian in terms of string coordinate fields. Let us start with kinetic term $\mathcal{L}_{kin}$. Contribution of AdS_5 part in $\mathcal{L}_{kin}$ has manifest $SL(2,C)\times SU(4)$ invariance from the very beginning. In order to transform contribution of S^5 part into desired form we notice that $L^{\mathrm{A}'}$ given by (2.9) and (3.29) can be cast into the form

$$L^{\mathrm{A}'} = e_{\mathcal{A}}^{\mathrm{A}'}(dy^{\mathcal{A}} - d\theta_{i}^{\mathsf{a}}(\mathcal{V}^{\mathcal{A}})^{i}{}_{j}\eta_{\mathsf{a}}^{j} + \eta_{i}^{\dot{\mathsf{a}}}(\mathcal{V}^{\mathcal{A}})^{i}{}_{j}d\theta_{\dot{\mathsf{a}}}^{j} + \mathrm{i}\mathsf{L}_{P}^{\mathsf{a}\dot{\mathsf{b}}}\eta_{\dot{\mathsf{b}}i}(\mathcal{V}^{\mathcal{A}})^{i}{}_{j}\eta_{\mathsf{a}}^{j})\,, \tag{4.1}$$

where $e_{\mathcal{A}}^{\mathrm{A}'}$ are components of S^5 vielbein $e^{\mathrm{A}'} = e_{\mathcal{A}}^{\mathrm{A}'}dy^{\mathcal{A}}$:

$$G_{\mathcal{A}\mathcal{B}} = e_{\mathcal{A}}^{\mathrm{A}'}e_{\mathcal{B}}^{\mathrm{A}'}\,, \qquad e_{\mathcal{A}}^{\mathrm{A}'} = \frac{\sin|y|}{|y|}(\delta_{\mathcal{A}}^{\mathrm{A}'} - n_{\mathcal{A}}n^{\mathrm{A}'}) + n_{\mathcal{A}}n^{\mathrm{A}'}\,, \tag{4.2}$$

and $(\mathcal{V}^{\mathcal{A}})^{i}{}_{j}$ are the components of the Killing vectors $(\mathcal{V}^{\mathcal{A}})^{i}{}_{j}\partial_{y^{\mathcal{A}}}$ of S^5 $(\partial_{y^{\mathcal{A}}} = \partial/\partial y^{\mathcal{A}})$. The $G_{\mathcal{A}\mathcal{B}}$ is metric tensor on S^5 in the coordinates $y^{\mathcal{A}}$ defined by coset representative g_y

(3.25),(3.26). The $(\mathcal{V}^{\mathcal{A}})^i{}_j\partial_{y^{\mathcal{A}}}$ satisfy the $so(6)\simeq su(4)$ commutation relations (3.2) and may be written as

$$(\mathcal{V}^{\mathcal{A}})^i{}_j\partial_{y^{\mathcal{A}}} = \frac{1}{4}(\gamma^{\mathrm{A'B'}})^i{}_j V^{\mathrm{A'B'}} + \frac{\mathrm{i}}{2}(\gamma^{\mathrm{A'}})^i{}_j V^{\mathrm{A'}}\ , \tag{4.3}$$

where $V^{\mathrm{A'}}$ and $V^{\mathrm{A'B'}}$ correspond to the 5 translations and $SO(5)$ rotations respectively and are given by

$$V^{\mathrm{A'}} = \Big[|y|\cot|y|(\delta^{\mathrm{A'}\mathcal{A}} - n^{\mathrm{A'}}n^{\mathcal{A}}) + n^{\mathrm{A'}}n^{\mathcal{A}}\Big]\partial_{y^{\mathcal{A}}}\ , \qquad V^{\mathrm{A'B'}} = y^{\mathrm{A'}}\partial_{y^{\mathrm{B'}}} - y^{\mathrm{B'}}\partial_{y^{\mathrm{A'}}}\ . \tag{4.4}$$

They satisfy the $so(6)$ algebra commutation relations given in (B.1),(B.2). Here $\delta^{\mathrm{A'}\mathcal{A}}$ is Kronecker delta symbol and we use the conventions: $y^{\mathcal{A}} = \delta^{\mathcal{A}}_{\mathrm{A'}}y^{\mathrm{A'}}$, $n^{\mathcal{A}} = \delta^{\mathcal{A}}_{\mathrm{A'}}n^{\mathrm{A'}}$, $n^{\mathcal{A}} = n_{\mathcal{A}}$. Note that while deriving (4.1) we use the relation

$$(U^\dagger\gamma^{\mathrm{A'}}U)^i{}_j = -2\mathrm{i}e^{\mathrm{A'}}_{\mathcal{A}}(\mathcal{V}^{\mathcal{A}})^i{}_j\ . \tag{4.5}$$

The $L^{\mathrm{A'}}L^{\mathrm{A'}}$ can be put into the manifestly $SU(4)$ invariant form by changing the coordinates from $y^{\mathrm{A'}}$ to the $6d$ unit vector u^M ($M=1,...,6$):

$$u^{\mathrm{A'}} = n^{\mathrm{A'}}\sin|y|\,, \qquad u^6 = \cos|y|\,, \qquad u^M u^M = 1\,, \tag{4.6}$$

and noticing the following important relation

$$L^{\mathrm{A'}}L^{\mathrm{A'}} = Du^M Du^M\,, \tag{4.7}$$

where

$$Du^M = du^M - d\theta^{\mathsf{a}}_i(R^M)^i{}_j\eta^j_{\mathsf{a}} + \eta^{\dot{\mathsf{a}}}_i(R^M)^i{}_j d\theta^j_{\dot{\mathsf{a}}} + \mathrm{i}\mathsf{L}_P^{\mathsf{a}\dot{\mathsf{b}}}\eta_{\dot{\mathsf{b}}i}(R^M)^i{}_j\eta^j_{\mathsf{a}}\,, \tag{4.8}$$

$$R^M \equiv -\frac{1}{2}\rho^{MN}u^N\,. \tag{4.9}$$

To derive (4.7) we used the identities

$$G_{\mathcal{AB}}(\lambda\mathcal{V}^{\mathcal{A}}\theta)(\vartheta\mathcal{V}^{\mathcal{B}}\eta) = (\lambda R^M\theta)(\vartheta R^M\eta)\,, \qquad G_{\mathcal{AB}}(\lambda\mathcal{V}^{\mathcal{A}}\theta)dy^{\mathcal{B}} = (\lambda R^M\theta)du^M\,, \tag{4.10}$$

where expressions like $(\lambda\mathcal{V}^{\mathcal{A}}\theta)$ stand for $\lambda_i(\mathcal{V}^{\mathcal{A}})^i{}_j\theta^j$. Making use of these relations we get the following manifest $SL(2,C)\times SU(4)$ invariant representation for kinetic term

$$\mathcal{L}_{kin} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}(\hat{L}^{\mathsf{a}\dot{\mathsf{b}}}_\mu\hat{L}_{\nu\mathsf{a}\dot{\mathsf{b}}} + D_\mu\phi D_\nu\phi + D_\mu u^M D_\nu u^M)\,, \tag{4.11}$$

where

$$D\phi \equiv d\phi + \frac{1}{2}(\eta^{\mathsf{a}i}d\theta_{\mathsf{a}i} + \eta^{\dot{\mathsf{a}}}_i d\theta^i_{\dot{\mathsf{a}}})\,. \tag{4.12}$$

Note that by changing the coordinates from ϕ, u^M to Z^M defined by

$$Z^M = e^{-\phi}u^M\,, \tag{4.13}$$

we can cast the kinetic term (4.11) into another manifest $SL(2,C)\times SU(4)$ invariant form given in (1.2).

We proceed now to the WZ three form. To transform the WZ term into desired form we use the standard trick of rescaling $\eta\to\eta_t\equiv t\eta$ (note that we do not rescale θ)

$$\mathcal{H}=\mathcal{H}_t(t=1)\,,\qquad \mathcal{H}_t=\mathcal{H}(\eta_t)\,. \tag{4.14}$$

Then one has the obvious relation

$$\mathcal{H}=\mathcal{H}_{t=0}+\int_0^1 dt\,\partial_t\mathcal{H}_t\,. \tag{4.15}$$

Note that shifting $\eta\to t\eta$ in coset representative (3.23) and then setting $t=0$ corresponds to chosing S gauge, i.e. $\mathcal{H}_{t=0}$ is nothing but WZ three form in S gauge. Its contribution to $\mathcal{L}_{WZ}$ has been already find in [5,6,10] (note we use convention of [10], where the details of deriving $\mathcal{H}_{t=0}$ may be found) and is given by

$$f\mathcal{L}_{WZ}|_{t=0}=\mathrm{i}e^{\phi}d\theta^{\dot{\mathsf{a}}i}y_{ij}d\theta^j_{\dot{\mathsf{a}}}+h.c.\,, \tag{4.16}$$

where

$$y_{ij}\equiv\rho^M_{ij}u^M\,. \tag{4.17}$$

To derive (4.16) we used the relation $U^TC'=C'U$ and the representation for charge conjugation C'_{ij} matrix and $SO(5)$ Dirac γ matrices given by (A.10) which together with (3.38) gives the following important relation

$$(C'U^2)_{ij}=y_{ij}\,. \tag{4.18}$$

The relation (4.18) expresses the fact that though neither C' matrix or U matrix do not transform covariantly under $SU(4)$ their special combination (4.18) transforms covariantly under $SU(4)$. Now we consider contribution of the second term in (4.15). Making shift $\eta\to t\eta$ in (3.23) and using (2.15) we get the following equations

$$\partial_t L_K^{\mathsf{a}\dot{\mathsf{b}}}=-\mathrm{i}L_{Si}^{\dot{\mathsf{b}}}\tilde{\tilde{\eta}}^{\mathsf{a}i}-\mathrm{i}L_S^{\mathsf{a}i}\tilde{\tilde{\eta}}_i^{\dot{\mathsf{b}}}\,, \tag{4.19}$$

$$\partial_t L_D=\frac12 L_{Qi}^{\mathsf{a}}\tilde{\tilde{\eta}}^i_{\mathsf{a}}+h.c.\,,\qquad \partial_t L^i{}_j=L^{\mathsf{a}}_{Qj}\tilde{\tilde{\eta}}^i_{\mathsf{a}}-\frac14\delta^i_jL^{\mathsf{a}}_{Qn}\tilde{\tilde{\eta}}^n_{\mathsf{a}}-h.c.\,, \tag{4.20}$$

$$\partial_t L_Q^{\dot{\mathsf{a}}i}=\mathrm{i}L_P^{\mathsf{b}\dot{\mathsf{a}}}\tilde{\tilde{\eta}}^i_{\mathsf{b}}\,,\qquad \partial_t L_S^{\mathsf{a}i}=d\tilde{\tilde{\eta}}^{\mathsf{a}i}+\frac12 L_D\tilde{\tilde{\eta}}^{\mathsf{a}i}+\frac12 L^{\mathsf{a}}{}_{\mathsf{b}}\tilde{\tilde{\eta}}^{\mathsf{b}i}-L^i{}_j\tilde{\tilde{\eta}}^{\mathsf{a}j}\,, \tag{4.21}$$

where

$$\tilde{\tilde{\eta}}^{\mathsf{a}i}\equiv e^{-\phi/2}U^i{}_j\eta^{\mathsf{a}j}\,,\qquad \tilde{\tilde{\eta}}_{\mathsf{a}i}\equiv\eta_{\dot{\mathsf{a}}j}(U^{-1})^j{}_ie^{-\phi/2}\,, \tag{4.22}$$

and we assume that in expressions for Cartan 1-forms in (4.19)-(4.21) all η are shifted $\eta\to t\eta$. Hermitean conjugation in $\partial_tL^i{}_j$ in (4.20) should be supplemented by $i\leftrightarrow j$. Now making use of (2.17), formulas (4.19)-(4.21) and Maurer-Cartan equations (3.13)-(3.20) we find

$$\partial_t(f\mathcal{H}^q)=d\Big[-2\mathrm{i}\hat{L}^{\mathsf{a}\dot{\mathsf{b}}}\tilde{\tilde{\eta}}^i_{\mathsf{a}}C'_{ij}L^j_{Q\dot{\mathsf{b}}}-L_D\tilde{\tilde{\eta}}^{\mathsf{a}i}C'_{ij}L^j_{S\mathsf{a}}+\tilde{\tilde{\eta}}^{\mathsf{a}i}((C'L)_{ij}-(C'L)_{ji})L^j_{S\mathsf{a}}\Big]\,, \tag{4.23}$$

where $\mathcal{H}^q \equiv \mathcal{H}^q_{AdS_5} + \mathcal{H}^q_{S^5}$. Making use of formula (4.23) and the relation

$$(U^T(C'L)U)_{ij} - (i \leftrightarrow j) = Du^M \rho^M_{ij}\,, \tag{4.24}$$

where $(C'L)_{ij}$ is defined by (2.20) we get finally the desired manifest $SL(2,C) \times SU(4)$ representation for $\mathcal{L}_{WZ}$

$$f\mathcal{L}_{WZ} = \mathrm{i}e^{\phi} d\theta^{\dot{\mathsf{a}}i} y_{ij} d\theta^j_{\dot{\mathsf{a}}} + \int_0^1 dt[2\hat{L}^{\mathsf{a}\dot{\mathsf{b}}} \eta^i_{\mathsf{a}} y_{ij} \mathsf{L}^j_{Q\dot{\mathsf{b}}} - \mathrm{i}e^{-\phi} L_D \eta^{\mathsf{a}i} y_{ij} \mathsf{L}^j_{S\mathsf{a}} + \mathrm{i}e^{-\phi} \eta^{\mathsf{a}i} D y_{ij} \mathsf{L}^j_{S\mathsf{a}}\Big] + h.c. \tag{4.25}$$

where $Dy_{ij} = \rho^M_{ij} Du^M$ and Du^M is defined by (4.8). In terms of coordinates Z^M defined by (4.13) we can cast this $\mathcal{L}_{WZ}$ into the form given by (1.3).

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Appendix A Notation

In the main part of the paper we use the following conventions for the indices:

$A, B, C = 0, \ldots, 4$	$so(4,1)$ vector indices (AdS_5 tangent space indices)
$\mathrm{A}', \mathrm{B}', \mathrm{C}' = 1, \ldots, 5$	$so(5)$ vector indices (S^5 tangent space indices)
$a, b, c = 0, \ldots, 3$	boundary Minkowski space indices
$\mathcal{A}, \mathcal{B}, \mathcal{C} = 1, \ldots, 5$	S^5 coordinate space indices
$M, N, K, L = 1, \ldots, 6$	$so(6)$ vector indices
$i, j, k, n = 1, \ldots, 4$	$su(4)$ vector indices
$\alpha, \beta, \gamma = 1, \ldots, 4$	$so(4,1)$ spinor indices
$\mathsf{a}, \mathsf{b}, \mathsf{c};\ (\dot{\mathsf{a}}, \dot{\mathsf{b}}, \dot{\mathsf{c}}) = 1, 2;\ (\dot{1}, \dot{2})$	$sl(2,C)$ spinor indices
$\mu, \nu = 0, 1$	world sheet coordinate indices

Note that we identify the $su(4)$ vector indices i, j, k, n with $so(5)$ spinor indices. We decompose x^a into the light-cone and 2 complex coordinates: $x^a = (x^+, x^-, x, \bar{x})$

$$x^{\pm} \equiv \frac{1}{\sqrt{2}}(x^3 \pm x^0)\,, \qquad x, \bar{x} = \frac{1}{\sqrt{2}}(x^1 \pm \mathrm{i}x^2)\,. \tag{A.1}$$

We suppress the flat space metric tensor $\eta_{ab} = (-,+,+,+)$ in scalar products, i.e. $A^a B^a \equiv \eta_{ab} A^a B^b$. The world-sheet Levi-Civita $\epsilon^{\mu\nu}$ is defined with $\epsilon^{01} = 1$.

We use the following decomposition of $so(4,1)$ Dirac γ^A and charge conjugation $C_{\alpha\beta}$ matrices in the $sl(2)$ basis

$$(\gamma^a)^\alpha{}_\beta = \begin{pmatrix} 0 & (\sigma^a)^{\mathsf{a}\dot{\mathsf{b}}} \\ \bar{\sigma}^a_{\dot{\mathsf{a}}\mathsf{b}} & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C_{\alpha\beta} = \begin{pmatrix} \epsilon_{\mathsf{ab}} & 0 \\ 0 & \epsilon^{\dot{\mathsf{a}}\dot{\mathsf{b}}} \end{pmatrix}, \tag{A.2}$$

where the matrices $(\sigma^a)^{\mathsf{a}\dot{\mathsf{a}}}$, $(\bar{\sigma}^a)_{\dot{\mathsf{a}}\mathsf{a}}$ are related to Pauli matrices in the standard way

$$(\sigma^a)^{\mathsf{a}\dot{\mathsf{a}}} = (1, \sigma^1, \sigma^2, \sigma^3), \qquad (\bar{\sigma}^a)_{\dot{\mathsf{a}}\mathsf{a}} = (-1, \sigma^1, \sigma^2, \sigma^3). \tag{A.3}$$

Note that $\bar{\sigma}^a_{\dot{\mathsf{a}}\mathsf{b}} = \sigma^a_{\mathsf{b}\dot{\mathsf{a}}} = \sigma^{a*}_{\mathsf{a}\dot{\mathsf{b}}}$ where $\sigma^a_{\mathsf{a}\dot{\mathsf{a}}} \equiv (\sigma^a)^{\mathsf{b}\dot{\mathsf{b}}}\epsilon_{\mathsf{ba}}\epsilon_{\dot{\mathsf{b}}\dot{\mathsf{a}}}$. Sometimes we use the relation

$$(\sigma^a)^{\mathsf{a}\dot{\mathsf{a}}}(\sigma^a)^{\mathsf{b}\dot{\mathsf{b}}} = 2\epsilon^{\mathsf{ab}}\epsilon^{\dot{\mathsf{a}}\dot{\mathsf{b}}}. \tag{A.4}$$

We use the following conventions for the $sl(2)$ indices: $\epsilon_{12} = \epsilon^{12} = -\epsilon_{\dot{1}\dot{2}} = -\epsilon^{\dot{1}\dot{2}} = 1$,

$$\psi^{\mathsf{a}} = \epsilon^{\mathsf{ab}}\psi_{\mathsf{b}}, \quad \psi_{\mathsf{a}} = \psi^{\mathsf{b}}\epsilon_{\mathsf{ba}}, \qquad \psi^{\dot{\mathsf{a}}} = \epsilon^{\dot{\mathsf{a}}\dot{\mathsf{b}}}\psi_{\dot{\mathsf{b}}}, \quad \psi_{\dot{\mathsf{a}}} = \psi^{\dot{\mathsf{b}}}\epsilon_{\dot{\mathsf{b}}\dot{\mathsf{a}}}. \tag{A.5}$$

Note that $(\epsilon^{\mathsf{ab}})^* = -\epsilon^{\dot{\mathsf{a}}\dot{\mathsf{b}}}$. We use the notation

$$(\sigma^{ab})_{\mathsf{ab}} = (\sigma^{ab})^{\mathsf{c}}{}_{\mathsf{b}}\epsilon_{\mathsf{ca}}, \qquad (\bar{\sigma}^{ab})_{\dot{\mathsf{a}}\dot{\mathsf{b}}} = (\bar{\sigma}^{ab})_{\dot{\mathsf{a}}}{}^{\dot{\mathsf{c}}}\epsilon_{\dot{\mathsf{c}}\dot{\mathsf{b}}}, \tag{A.6}$$

where

$$(\sigma^{ab})^{\mathsf{a}}{}_{\mathsf{b}} \equiv \frac{1}{2}(\sigma^a)^{\mathsf{a}\dot{\mathsf{c}}}(\bar{\sigma}^b)_{\dot{\mathsf{c}}\mathsf{b}} - (a \leftrightarrow b), \qquad (\bar{\sigma}^{ab})_{\dot{\mathsf{a}}}{}^{\dot{\mathsf{b}}} \equiv \frac{1}{2}(\bar{\sigma}^a)_{\dot{\mathsf{a}}\mathsf{c}}(\sigma^b)^{\mathsf{c}\dot{\mathsf{b}}} - (a \leftrightarrow b). \tag{A.7}$$

Note that $(\sigma^{ab})_{\mathsf{ab}} = (\sigma^{ab})_{\mathsf{ba}}$, $(\sigma^{ab})^*_{\mathsf{ab}} = (\bar{\sigma}^{ab})_{\dot{\mathsf{a}}\dot{\mathsf{b}}}$.

The six matrices ρ^M_{ij} represent the $SO(6)$ Dirac matrices γ^M in the chiral representation, i.e.

$$\gamma^M = \begin{pmatrix} 0 & (\rho^M)^{ij} \\ \rho^M_{ij} & 0 \end{pmatrix}, \tag{A.8}$$

$$(\rho^M)^{ik}\rho^N_{kj} + (\rho^N)^{ik}\rho^M_{kj} = 2\delta^{MN}\delta^i_j, \qquad \rho^M_{ij} = -\rho^M_{ji}, \qquad (\rho^M)^{ij} \equiv -(\rho^M_{ij})^*. \tag{A.9}$$

The $SO(5)$ Dirac and charge conjugation matrices can be expressed in terms of the ρ^M matrices as follows

$$(\gamma^{\mathrm{A}'})^i{}_j = \mathrm{i}(\rho^{\mathrm{A}'})^{ik}\rho^6_{kj}, \qquad C'_{ij} = \rho^6_{ij}. \tag{A.10}$$

The ρ^M matrices satisfy the identities

$$\rho^M_{ij} = \frac{1}{2}\epsilon_{ijkn}(\rho^M)^{kn}, \qquad \rho^M_{ij}(\rho^M)^{kn} = 2(\delta^n_i\delta^k_j - \delta^k_i\delta^n_j). \tag{A.11}$$

The matrices ρ^{MN} are defined by

$$(\rho^{MN})^i{}_j \equiv \frac{1}{2}(\rho^M)^{ik}\rho^N_{kj} - (M \leftrightarrow N), \tag{A.12}$$

so that

$$(\rho^{MN})^i{}_j(\rho^{MN})^k{}_n = 2\delta^i_j\delta^k_n - 8\delta^i_n\delta^k_j. \tag{A.13}$$

Appendix B Transformation of $psu(2,2|4)$ superalgebra from $so(4,1)\oplus so(5)$ to $sl(2,C)\oplus su(4)$ basis

We start with the commutation relations of $psu(2,2|4)$ superalgebra in $so(4,1)\oplus so(5)$ basis given in [2]

$$[\hat{P}_A, \hat{P}_B] = \hat{J}_{AB}\,, \qquad [P_{\mathrm{A}'}, P_{\mathrm{B}'}] = -J_{\mathrm{A}'\mathrm{B}'}\,, \tag{B.1}$$

$$[\hat{J}^{AB}, \hat{J}^{CE}] = \eta^{BC}\hat{J}^{AE} + 3\ \text{terms}, \qquad [J^{\mathrm{A}'\mathrm{B}'}, J^{\mathrm{C}'\mathrm{E}'}] = \eta^{\mathrm{B}'\mathrm{C}'}J^{\mathrm{A}'\mathrm{E}'} + 3\ \text{terms}\,, \tag{B.2}$$

$$[Q_{_I}, \hat{P}_A] = -\frac{\mathrm{i}}{2}\epsilon_{_{IJ}}Q_{_J}\gamma_A\,, \qquad [Q_{_I}, \hat{J}_{AB}] = -\frac{1}{2}Q_{_I}\gamma_{AB}\,, \tag{B.3}$$

$$[Q_{_I}, P_{\mathrm{A}'}] = \frac{1}{2}\epsilon_{_{IJ}}Q_{_J}\gamma_{\mathrm{A}'}\,, \qquad [Q_{_I}, J_{\mathrm{A}'\mathrm{B}'}] = -\frac{1}{2}Q_{_I}\gamma_{\mathrm{A}'\mathrm{B}'}\,, \tag{B.4}$$

$$\begin{aligned}\{Q_{\alpha i I}, Q_{\beta j J}\} &= \delta_{_{IJ}}[-2\mathrm{i}C'_{ij}(C\gamma^A)_{\alpha\beta}\hat{P}_A + 2C_{\alpha\beta}(C'\gamma^{\mathrm{A}'})_{ij}P_{\mathrm{A}'}] \\ &+ \epsilon_{_{IJ}}[C'_{ij}(C\gamma^{AB})_{\alpha\beta}\hat{J}_{AB} - C_{\alpha\beta}(C'\gamma^{\mathrm{A}'\mathrm{B}'})_{ij}J_{\mathrm{A}'\mathrm{B}'}]\,.\end{aligned} \tag{B.5}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\gamma_{AB} = \frac{1}{2}\gamma_A\gamma_B - (A \leftrightarrow B)$. Unless otherwise specified, we use the notation Q^I for $Q^{I\alpha i}$ and Q_I for $Q_{I\alpha i}$, where

$$Q_{I\alpha i} \equiv Q^{J\beta j}\delta_{JI}C_{\beta\alpha}C'_{ji}\,. \tag{B.6}$$

Hermitean conjugation rules in this basis are

$$\hat{P}^\dagger_A = -\hat{P}_A\,, \qquad P^\dagger_{\mathrm{A}'} = -P_{\mathrm{A}'}\,, \qquad \hat{J}^\dagger_{AB} = -\hat{J}_{AB}\,, \qquad J^\dagger_{\mathrm{A}'\mathrm{B}'} = -J_{\mathrm{A}'\mathrm{B}'}\,, \tag{B.7}$$

$$(Q^{I\beta i})^\dagger(\gamma^0)_\alpha^\beta = -Q^{I\beta j}C_{\beta\alpha}C'_{ji}\,. \tag{B.8}$$

The transformation of the bosonic generators into the conformal algebra basis are given by formulas (2.6) Let us describe transformation of fermionic generators. First we introduce the new "charged" super-generators

$$Q^q \equiv \frac{1}{\sqrt{2}}(Q^1 + \mathrm{i}Q^2)\,, \qquad Q^{\bar{q}} \equiv \frac{1}{\sqrt{2}}(Q^1 - \mathrm{i}Q^2)\,. \tag{B.9}$$

We shall use the simplified notation

$$Q^{\alpha i} \equiv -Q^{q\alpha i}\,, \qquad Q_{\alpha i} \equiv Q_{q\alpha i}\,. \tag{B.10}$$

Then the non-vanishing values of δ_{IJ} (ϵ_{IJ}, $\epsilon_{12} = 1$) become replaced by $\delta_{q\bar{q}} = 1$ ($\epsilon_{q\bar{q}} = \mathrm{i}$) and the Majorana condition takes the form $(Q^{\beta i})^\dagger(\gamma^0)_\alpha^\beta = Q_{\alpha i}$. The relation (B.6) and (B.10) give

$$Q_{\bar{q}\alpha i} = -Q^{\beta j}C_{\beta\alpha}C'_{ji}\,. \tag{B.11}$$

We then decompose the supercharges in the $sl(2) \oplus su(4)$ basis

$$Q^{\alpha i} = \begin{pmatrix} 2\mathrm{i}v^{-1}Q^{\mathsf{a}i} \\ 2vS^i_{\dot{\mathsf{a}}} \end{pmatrix} , \qquad Q_{\alpha i} = (2vS_{\mathsf{a}i}, -2\mathrm{i}v^{-1}Q^{\dot{\mathsf{a}}}_i) , \qquad v \equiv 2^{1/4} . \tag{B.12}$$

In terms of these new supercharges the commutation relations take the form given in the Section 3.

Now let us consider the fermionic 1-forms. They satisfy hermitean conjugation rule

$$(L^{I\beta i})^\dagger (\gamma^0)_\alpha^\beta = L^{I\beta j} C_{\beta\alpha} C'_{ji}. \tag{B.13}$$

and we use the notation $L_{I\alpha i} \equiv L^{J\beta j}\delta_{JI} C_{\beta\alpha} C'_{ji}$. Let us define

$$L^q \equiv \frac{1}{\sqrt{2}}(L^1 + \mathrm{i}L^2) , \qquad L^{\bar{q}} \equiv \frac{1}{\sqrt{2}}(L^1 - \mathrm{i}L^2) , \tag{B.14}$$

and introduce the notation $L^{\alpha i} = L^{q\alpha i}$, $L_{\alpha i} = L_{q\alpha i}$. These relations give

$$L_{\alpha i} = L^{\bar{q}\beta j} C_{\beta\alpha} C'_{ji} . \tag{B.15}$$

We use then the following decomposition into $sl(2) \oplus su(4)$ Cartan 1-forms

$$L^{\alpha i} = \frac{1}{2}\begin{pmatrix} v^{-1}L_S^{\mathsf{a}i} \\ \mathrm{i}vL^i_{Q\,\dot{\mathsf{a}}} \end{pmatrix} , \qquad L_{\alpha i} = \frac{1}{2}(-\mathrm{i}vL_{Q\,\mathsf{a}i},\, v^{-1}L^{\dot{\mathsf{a}}}_{Si}) . \tag{B.16}$$

Hermitean conjugation rules for the new Cartan 1-forms in $sl(2) \oplus su(4)$ basis take then the form given in (3.22). The above relations lead to the decomposition $L^{I\alpha i}Q_{I\alpha i} = L^{\alpha i}Q_{\alpha i} - L_{\alpha i}Q^{\alpha i}$ which in terms of $sl(2) \oplus su(4)$ notation takes the form given in the second line in (2.15).

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